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Physical Equivalence of Theories

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Abstract

A means of distinguishing between equivalent and different (inequivalent) theories is presented. It is applied to distinguish between the Brans–Dicke theory and general relativity. It is shown that an infinite number of Brans–Dicke-type extensions can be constructed for all field theories.

1. *Introduction*

Numerous theories have recently been presented that differ from previously known (and accepted) theories in some way but are not very different. It may often be necessary to distinguish between such theories explicitly so as to be able to determine whether there is some essential difference between them or the difference can basically be ignored. For this purpose we shall define what we mean by saying that two theories are equivalent or inequivalent. We can find a finite preferred subclass of the equivalence class of theories given by our definition.

We shall apply the procedure developed to state general relativity in a form that makes it obvious that it is an element of the preferred set of its equivalence class. The procedure could be applied to construct theories equivalent to other well-known theories, but we shall restrict our attention mainly to general relativity, as it has suffered the most from having too many theories constructed that are physically equivalent to it.

We shall then proceed to consider nearly equivalent theories. It will be seen that the Brans-Dicke theory, though different from general relativity (to the extent of being inequivalent) is nearly equivalent to it in this sense. We

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shall see that an infinite number of nearly equivalent extensions can be constructed for *any* field theory. However, such extensions lose predictive power and there may therefore be doubt about the theories remaining scientific.

We contend that such theories as the Brans–Dicke theory, or other theories equivalent or nearly equivalent to general relativity, are not needed at present, and their construction should be avoided unless they purport to make some point in the theory manifest.

2. *Physical Theories*

When we talk of "theories" here we shall always mean "scientific theories" in the sense of Popper, i.e., a finite set of axioms that is used to make predictions that may, in principle, be found to be inconsistent with experiment. When the axioms are not able to explain the experimental evidence the theory is said to have been falsified. The requirement that the set of axioms be finite is due to the fact that the axioms must be actually *enumerated* (Qadir, 1975).

It will be shown later that even a finite set of axioms need not give falsifiable results if we appropriately choose our means of measurement. This, in effect, alters the axioms without explicitly appearing to do so. To avoid this problem we shall define a *physical theory* as a finite set of axioms together with a specified means of measuring the fundamental quantities.

Applying Gödel's theorem to a scientific theory we see that it will not be *complete* (Qadir, 1975) in the sense that there will exist axioms that can be taken to be or taken not to be consistent with the given scientific theory. However, if we require that procedures of measurement also be prescribed, i.e., that we have a physical theory, it may be possible to construct a complete theory.

It should be noted that a whole system of concepts is required to give meaning to the set of axioms forming a physical theory. This set need not form a scientific theory, not even being reducible to a finite set of axioms in general. We shall call this system the *subscientific theory* (Qadir, 1975). In general two scientific theories will have different subscientific theories. In order to be able to compare them we shall require that any two physical theories we want to test for equivalence have the same subscientific theories. We achieve this by taking the differences explicitly into the statement of the physical theories.

3. *Physical Equivalence of Physical Theories*

We shall call two physical theories *equivalent* if the set of measurement definitions is the same and the set of axioms can be reduced one into the other. This definition obviously restricts what we mean by equivalent theories too strongly. We shall, therefore, define a weaker form of equivalence.

Two physical theories will be said to be *physically equivalent* if there exists a one-to-one correspondence between the two definitions of measurement such that all the predictions made by each theory get converted to those made by the other theory. Thus two theories will be physically equivalent if and only if no experiment can falsify one without falsifying the other. We can obviously construct equivalence classes of physically equivalent theories.

In the equivalence class of a given physical theory there will be some that have a minimum number of axioms to form the scientific theory. We shall call this set of physical theories the *ideal set* of physical theories equivalent to the given physical theory. An element of this set will be called an *ideal physical theory*. In principle it should be easiest to show that two physical theories are physically equivalent (or inequivalent) by showing that two ideal physical theories of the equivalence classes of the two physical theories are physically equivalent (or inequivalent). Note that by "inequivalent theories" we mean two theories that are not physically equivalent.

4. *Application to Relativity*

An attempt has been made to state an ideal physical theory physically equivalent to the theory of special relativity [causal relativity (Qadir, unpublished)]. There it was found that one axiom was sufficient to derive the theory (which is physically equivalent to special relativity with tachyons not being allowed). In that case equivalence (and physical equivalence) becomes trivially obvious. This is not the case in general relativity.

It has been previously shown that there does exist at least one theory physically equivalent to general relativity in conformally flat space (Qadir, 1976). The extension to curved space follows naturally. There the difference between general relativity and the theory of reciprocity is in the choice of the zero acceleration frame being of the distant stars or arbitrary. When choosing the frame to be arbitrary we imposed a corresponding transformation of time measurement. We could equally well have imposed a transformation of spacial interval measurements to achieve the same type of results. Alternatively we could have fixed the frame of reference of the Earth as the frame of zero acceleration and then bring in the so-called "fictitious forces." Clearly there will be a difference in the "physical laws" of the theories, but their predictions will agree.

What, then, is the essence of general relativity? To determine this we need to state it as an ideal physical theory with the usual subscientific theory already assumed. One requirement is that space-time form a four-real-dimensional manifold. The second is that Einstein's equations

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = -KT_{ab} \tag{4.1}$$

hold, where T_{ab} is the stress energy tensor and is given, g_{ab} is the metric tensor that is to be determined by solving equations (4.1), R_{ab} is the Ricci tensor, and $R(= g^{cd}R_{cd})$ is the Ricci scalar, K being a constant to be determined by experiment. Time is to be measured by "atomic clocks," distance by light signals, and masses by inertial effects. A third requirement is that mass measured by gravitational and inertial effects be the same. Finally, the fourth assumption (used in causal relativity) is of causality.

5. *Nearly Physically Equivalent Physical Theories*

We had previously defined two physical theories to be physically equivalent if every conceivable experiment that falsifies one would falsify the other. Even this restriction is too strong for many purposes. We shall define two physical theories to be *nearly physically equivalent* if every performable experiment that falsifies one would falsify the other. Thus it depends on the technological experimental situation.

We can always construct physical field theories that fit the experimental facts as well as some well-accepted physical field theory while being inequivalent to it by constructing a nearly physically equivalent theory. This would be best illustrated by an example. We shall take classical electromagnetism since it is one of the simplest examples that can be constructed.

Let us restrict the statement of the theory by absorbing the rest of the axioms and definitions into the subscience. We can then write the theory as Maxwell's field equations

$$\nabla_a F^{ab} = J^b \tag{5.1}$$

$$\nabla_{[a} F_{bc]} = 0 \tag{5.2}$$

together with the definition of the Maxwell field tensor in terms of the four-vector potential

$$F_{ab} = 2\nabla_{[a} A_{b]} \tag{5.3}$$

Clearly the theory allows changes of the gauge transformation type

$$A_a \to \tilde{A}_a = A_a + \nabla_a \psi \tag{5.4}$$

However, a physically inequivalent theory can be constructed by taking

$$A_a \to \tilde{A}_a = A_a + \varphi \nabla_a \psi \tag{5.5}$$

where φ and ψ are scalar fields. Therefore we have

$$F_{ab} \to \tilde{F}_{ab} = F_{ab} + 2(\nabla_{[a}\varphi)(\nabla_{b]}\psi) \tag{5.6}$$

We can make the theories nearly physically equivalent by having φ and ψ vary sufficiently slowly in all directions, or by making them sufficiently close to each other. It is obvious that we can construct an infinite number of such physical theories nearly equivalent to classical electromagnetism. There being an infinite number of such extensions to all field theories, such extensions cannot be interesting in themselves.

6. *Near Equivalence and General Relativity*

The experimental situation for general relativity is highly satisfactory (Nordvedt, 1975), i.e., the theory fits experiment within experimental errors. There is, therefore, no room for a theory that is not nearly equivalent to general relativity. Why, then, should we have other theories? There can be various reasons, of which two seem useful: One is that some point has not been made manifest in general relativity that could clarify a particular situation, e.g., the theory of reciprocity (Khan, 1968, 1972). The other would be to construct a more powerful physical theory, i.e., one in which with the same, or fewer, assumptions more predictions can be made. In the first case we have physical equivalence, but in the second we do not even have near physical equivalence.

Let us first consider the theory of general relativity with a cosmological constant, to determine whether it falls into either of the above categories. The only difference from general relativity is that equation (4.1) is replaced by

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = -KT_{ab} \tag{6.1}$$

where K and Λ are both to be determined by experiment. Since general relativity is satisfactory, equation (6.1) gives a nearly physically equivalent theory and is therefore not in either of the above categories. The sole function that Λ can perform is to allow more experimental facts to be fitted into the theory, and it thus gives a weaker physical theory, i.e., one with less predictive power.

Let us now consider the Brans-Dicke theory (Brans and Dicke, 1961). This can be stated by replacing equation (4.1) of general relativity by

$$\begin{aligned} R_{ab} - \tfrac{1}{2}Rg_{ab} = {} & -KT_{ab} + (\omega/\varphi^2)(\nabla_a\varphi\nabla_b\varphi - \tfrac{1}{2}g_{ab}\nabla_c\varphi\nabla^c\varphi) \\ & + (1/\varphi)(\nabla_a\nabla_b\varphi - g_{ab}\nabla_c\nabla^c\varphi) \end{aligned} \tag{6.2}$$

φ being a scalar field having gravitational effects and ω a coupling parameter, and the additional statement of Mach's principle that every change in the matter distribution of the Universe must have a unique effect on a local observer. This gives a varying scalar field instead of a gravitational constant, given by

$$G = G_0/[1 + 2GM/(3 + 2\omega)c^2r] \tag{6.3}$$

where G_0 is the usual gravitational constant, M the mass, c the speed of light, and and r the radial position, in the weak field case. For sufficiently large ω equation (6.3) will reduce to usual general relativity and will therefore give a theory physically equivalent to general relativity, in spite of the fact that Mach's principle was assumed. Thus Mach's principle as interpreted by Brans-Dicke is fully incorporated into equation (6.2). In fact Brans and Dicke have determined the lower limit of ω by making their theory nearly physically equivalent to general relativity. They had originally fixed the lower limit of ω as 6. It has since moved to 200 (Nordvedt, 1975). Thus again the Brans-Dicke theory is not in either of the above-mentioned categories.

We could go on constructing such theories, but as we essentially require that the new theory be nearly physically equivalent to general relativity, the work seems pointless. This does not, of course, mean that no attempts to construct more powerful theories should be made, but certainly it is pointless to construct physical theories that are weaker than general relativity (like the Brans–Dicke theory) till forced to by the experimental evidence.

It might also be noted, in passing, that Mach's principle can simply be interpreted as saying that the origin of mass (inertial and gravitational) is due to its interaction with the rest of the matter in the Universe. Thus an attempt to deal with mass as originating from interaction with gravitons might be a fruitful search for a field theoretic approach to general relativity. In this way of thinking, the bare mass is zero, but owing to the divergence of the higher-order self-interaction Feynman graphs an effective mass is produced. Here again an extra parameter is introduced in the renormalization, but there is hope that it may cancel the other quantum-electrodynamic divergences (Isham *et al.*, 1972).

7. *Conclusion and Remarks*

We have seen that we can define equivalence for physical theories in a useful way, in that it leads to a deeper understanding of the theories and helps us to choose more useful theories for our purposes. It would be very useful if there were a universal set of transformations to enable us to convert from one physical theory to another physically equivalent theory. As we require a very general sort of transformation that converts one definition of measurements to another, we might consider the "units transformations" of Dicke (Dicke, 1962). Let us consider this in greater detail.

With Dicke's units transformations we can convert a conformally curved space into a conformally flat (and hence a flat) space. If this could be applied to general relativity it should give us special relativity (with the added assumption that space-time is a four-real-dimensional manifold). Of course, the definition of measurement in the flat space-time is not operationally determinable by *any* means. Thus the theory ceases to be a physical theory. Now any scientific theory that is not a physical theory cannot have any definable means of measurement. Since we are requiring measurements to be made, the theory will not even be a scientific theory. This is obvious if we notice that every measurement has to be separately defined (at each point), so the theory can never be falsified. Thus we cannot use Dicke's units transformations generally. It may be possible to find suitable restrictions of the units transformations that would yield appropriate transformations between physically equivalent theories.

We saw that we could also define nearly physically equivalent theories in a useful way, in that it helped us to decide what direction should not be taken in further searches for new theories. It does not, of course, tell us which direction should be taken—but that could scarcely be expected.

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